

A NEW FORMULA FOR THE GENERATING FUNCTION OF THE NUMBERS OF SIMPLE GRAPHS

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ABSTRACT. By using an approach of the invariant theory we obtain a new formula for the ordinary generating function of the numbers of the simple graphs with n nodes.

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1. Introduction. Let $a_{n,i}$ be the number of simple graphs with n vertices and k edges. Let

$$g_n(z) = \sum_{i=0}^m a_{n,i} z^i, m = \binom{n}{2},$$

be the ordinary generating function for the sequence $\{a_{n,i}\}$, the OIES sequence A008406. For the small n we have

$$\begin{aligned} g_1(z) &= 1, \\ g_2(z) &= 1 + z, \\ g_3(z) &= 1 + z + z^2 + z^3, \\ g_4(z) &= 1 + z + 2z^2 + 3z^3 + 2z^4 + z^5 + z^6. \end{aligned}$$

An expression for $g_n(z)$ in terms of group cycle index was found by Harary in [1]. The result is based on Polya's efficient method for counting graphs, see [2] and [3]. Let G be a permutation group acting on the set $[n] := \{1, 2, \dots, n\}$. It is well known that each permutation α in G can be written uniquely as a product of disjoint cycles. Let $j_i(\alpha)$ be the number of cycles of length i , $1 \leq i \leq n$ in the disjoint cycle decomposition of α . Then the cycle index of G denoted $Z(G, s_1, s_2, \dots, s_n)$, is the polynomial in the variables s_1, s_2, \dots, s_n defined by

$$Z(G, s_1, s_2, \dots, s_n) = \frac{1}{|G|} \sum_{\alpha \in G} \prod_{i=0}^n s_i^{j_i(\alpha)}.$$

Denote by $[n]^{(2)}$ the set of 2-subsets of $[n]$. Let S_n be a permutation group on the set $[n]$. The pair group of S_n , denoted $S_n^{(2)}$ is the permutation group induced by S_n which acts on $[n]^{(2)}$. Specifically, each permutation $\sigma \in S_n$ induces a permutation $\sigma' \in S_n^{(2)}$ such that for every element $\{i, j\} \in [n]^{(2)}$ we have $\sigma'\{i, j\} = \{\sigma i, \sigma j\}$.

In [1] F. Harary proved that the generating function $g_n(z)$ is determined by substituting $1 + z^k$ for each variable s_k in the cycle index $Z(S_n^{(2)}, s_1, s_2, \dots, s_n)$. Symbolically

$$g_n(z) = Z(S_n^{(2)}, 1 + z),$$

where

$$Z(S_n^{(2)}) = \frac{1}{n!} \sum_{j_1+2j_2+\dots+nj_n=n} \frac{n!}{\prod_{k=1}^n k^{j_k} j_k!} \prod_k s_{2k+1}^{kj_{2k+1}} \prod_k (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{k \binom{j_k}{2}} \prod_{r < t} s_{[r,t]}^{(r,t)j_r j_t}.$$

In the paper by an approach of the invariant theory we derive another formula for the generating function $g_n(z)$.

Let \mathcal{V}_n be a vector space of weighted graphs on n vertices over the field \mathbb{K} , $\dim \mathcal{V}_n = m$. The group $S_n^{(2)}$ acts naturally on \mathcal{V}_n by permutations of the basic vectors. Consider the corresponding action of the group $S_n^{(2)}$ on the algebra of polynomial functions $\mathbb{K}[\mathcal{V}_n]$ and let $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ be the corresponding algebra of invariants. Let \mathcal{V}_n^0 be the set of simple graphs. The corresponding algebra of invariants $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$ is a finite-dimensional vector space and can be expanded into the direct sum of its subspaces:

$$\mathbb{K}[\mathcal{V}_n^0] = (\mathbb{K}[\mathcal{V}_n^0])_0 + (\mathbb{K}[\mathcal{V}_n^0])_1 + \dots + (\mathbb{K}[\mathcal{V}_n^0])_m.$$

In the paper we have proved that $\dim(\mathbb{K}[\mathcal{V}_n^0])_i = a_{n,i}$. Thus the generating function $g_n(z)$ coincides with the Poincaré series $\mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}; z)$ of the algebra invariants $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$.

Let us identify the elements of the group $S_n^{(2)}$ with permutation $m \times m$ matrices and denote $\mathbf{1}_m$ the identity $m \times m$ matrix. In the paper we offer the following formula for the generating function $g_n(z)$:

$$g_n(z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(\mathbf{1}_m - \alpha \cdot z^2)}{\det(\mathbf{1}_m - \alpha \cdot z)}.$$

Also for the generating function $m_n(z)$ of multigraphs on n vertices we prove that

$$m_n(z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{1}{\det(\mathbf{1}_m - \alpha \cdot z)}.$$

2. Algebra of invariants of simple graphs. Let \mathbb{K} be a field of characteristic zero. Denote by \mathcal{V}_n the set of undirected graphs on the vertices $\{1, \dots, n\}$ and whose edges are weighted in \mathbb{K} . A simple graph is a graph with weights in $\{0, 1\}$ and a multigraph is a graph with weights in \mathbb{K} . For any pair $\{i, j\}$ let $\mathbf{e}_{\{i,j\}}$ be the simple graph with one single edge $\{i, j\}$ and let $g_{\{i,j\}} \mathbf{e}_{\{i,j\}}$ be the graph with one single edge $\{i, j\}$ and with the weight $g_{\{i,j\}} \in \mathbb{K}$. The set \mathcal{V}_n is the vector space with the basis $\langle \mathbf{e}_{\{1,2\}}, \mathbf{e}_{\{1,3\}}, \dots, \mathbf{e}_{\{n-1,n\}} \rangle$ of dimension $m = \binom{n}{2}$. Indeed, any graph can be written uniquely as a sum $\sum g_{\{i,j\}} \mathbf{e}_{\{i,j\}}$. Let \mathcal{V}_n^* be the dual space with dual basis generated by the linear functions $x_{\{i,j\}}$ for which $x_{\{i,j\}}(\mathbf{e}_{\{k,l\}}) = \delta_{ik} \delta_{jl}$. The symmetric group S_n acts on \mathcal{V}_n and on \mathcal{V}_n^* by

$$\sigma \mathbf{e}_{\{i,j\}} = \mathbf{e}_{\{\sigma(i), \sigma(j)\}}, \sigma^{-1} x_{\{i,j\}} = x_{\{\sigma(i), \sigma(j)\}}.$$

Let us expand the action on the algebra of polynomial functions $\mathbb{K}[\mathcal{V}_n] = \mathbb{K}[\{x_{\{i,j\}}\}]$.

We say that a polynomial function $f \in \mathbb{K}[\{x_{\{i,j\}}\}]$ of m variables $x_{\{i,j\}}$ is a S_n -invariant if $\sigma f = f$ for all $\sigma \in S_n$. The S_n -invariants form a subalgebra $\mathbb{K}[\mathcal{V}_n]^{S_n}$ which is called the algebra of invariants of the vector space of the weighted graphs in n vertices. It is clear that there is an isomorphism $\mathbb{K}[\mathcal{V}_n]^{S_n} \cong \mathbb{K}[\mathcal{V}_n^0]^{S_n}$.

For convenience, we introduce a new set of variables:

$$\{x_1, x_2, \dots, x_m\} = \{x_{\{1,2\}}, x_{\{1,3\}}, \dots, x_{\{n-1,n\}}\}.$$

Then the action of S_n on the set $\{x_{\{1,2\}}, x_{\{1,3\}}, \dots, x_{\{n-1,n\}}\}$ induces its action of the pair group $S_n^{(2)}$ on the set $\{x_1, x_2, \dots, x_m\}$. We have

$$\mathbb{K}[x_{\{i,j\}}]^{S_n} \cong \mathbb{K}[x_1, x_2, \dots, x_m]^{S_n^{(2)}}.$$

In this notation any graph can be written in the way

$$g_1 \mathbf{e}_1 + g_2 \mathbf{e}_2 + \dots + g_m \mathbf{e}_m, g_i \in \mathbb{K},$$

where \mathbf{e}_s is the edge which connect the vertices $\{i', j'\}$ if the pair $\{i', j'\}$ has got the number s . Thus, in this case the old variable $x_{\{i', j'\}}$ corresponds to the new variable x_s .

Since for the simple graphs all its weights are 0, 1 then the reduction of the algebra $\mathbb{K}[\mathcal{V}_n]^{S_n}$ on the set of simple graphs has a simple structure.

Denote by \mathcal{V}_n^0 the set of all simple graphs on n vertices:

$$\mathcal{V}_n^0 = \left\{ \sum_{i=1}^m g_i \mathbf{e}_i \mid g_i \in \{0, 1\} \right\} \subset \mathcal{V}_n,$$

The corresponding subalgebra of polynomial function $\mathbb{K}[\mathcal{V}_n^0] \subset \mathbb{K}[\mathcal{V}_n]$ is generated by polynomial functions x_i which on every simple graph only take values 1 or 0.

Let us consider the ideal $I_m = (x_1^2 - x_1, x_2^2 - x_2, \dots, x_m^2 - x_m)$ in the algebra $\mathbb{K}[\mathcal{V}_n] = \mathbb{K}[x_1, x_2, \dots, x_m]$. The following statement holds:

Theorem 1.

$$(i) \quad \mathbb{K}[\mathcal{V}_n^0] \cong \mathbb{K}[x_1, x_2, \dots, x_m] / I_m,$$

$$(ii) \quad \mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}} \cong \mathbb{K}[x_1, x_2, \dots, x_m]^{S_n^{(2)}} / I_m.$$

Proof. On the ring of polynomial functions $\mathbb{K}[x_1, x_2, \dots, x_m]$ let us introduce a binary relation \sim : $f \sim g$ if $f = g$, considered as functions from $\{0, 1\}^m$ to \mathbb{K} . Obviously that $x_i^p \sim x_i$, for all $p \geq 1$. Define an endomorphism $\gamma : \mathbb{K}[\mathcal{V}_n] \rightarrow \mathbb{K}[\mathcal{V}_n^0]$ by the way:

$$\gamma(x_i^p) = x_i.$$

It is clear that the kernel of the endomorphism is exactly the ideal I_m . Then

$$\mathbb{K}[\mathcal{V}_n^0] \cong \mathbb{K}[x_1, x_2, \dots, x_m] / I_m.$$

Note that the algebra $\mathbb{K}[\mathcal{V}_n^0]$ is a finite dimensional vector space of the dimension 2^m with the basis

$$1, x_1, x_2, \dots, x_n, x_1 x_2, x_1 x_2, \dots, x_{m-1} x_m, \dots, x_1 x_2 \cdots x_m.$$

(ii) It is enough to prove that γ commutes with the action of the group $S_n^{(2)}$. Without loss of generality it is sufficient to check on the monomials. For arbitrary element $\sigma \in S_n^{(2)}$ and for arbitrary monomial $x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}$, $s \leq m$ we have

$$\begin{aligned} \gamma(\sigma^{-1}(x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})) &= \gamma(x_{\sigma(1)}^{k_1} x_{\sigma(2)}^{k_2} \cdots x_{\sigma(s)}^{k_s}) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(s)} = \\ &= \sigma^{-1}(x_1 x_2 \cdots x_s) = \sigma^{-1}(\gamma(x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})). \end{aligned}$$

□

If we know the algebra of invariants $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ then we are able to find the algebra of invariants $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$ of simple graphs. Indeed, if the invariants f_1, f_2, \dots, f_s generate the algebra $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ then the surjectivity of γ implies that the invariants $\gamma(f_1), \gamma(f_2), \dots, \gamma(f_s)$ generate the algebra $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$.

Example. Let us consider the case $n = 4$. The algebra of invariants $\mathbb{K}[\mathcal{V}_4]^{S_4^{(2)}}$ is well known, see [4], and its minimal generating system consists of the following 9 invariants:

$$\begin{aligned}
R(x_1) &= \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6), R(x_1^2) = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2), \\
R(x_1x_6) &= \frac{1}{3}(x_1x_6 + x_2x_5 + x_3x_4), R(x_1^3) = \frac{1}{6}(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3), \\
24R(x_1^2x_2) &= x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1^2x_4 + x_1^2x_5 + x_4^2x_1 + \\
&\quad + x_4^2x_5 + x_5^2x_1 + x_5^2x_4 + x_2^2x_4 + x_2^2x_6 + x_4^2x_2 + x_4^2x_6 + x_6^2x_2 + x_6^2x_4 + x_3^2x_5 + \\
&\quad + x_3^2x_6 + x_5^2x_3 + x_5^2x_6 + x_6^2x_3 + x_6^2x_5 \\
R(x_1x_2x_3) &= \frac{1}{4}(x_1x_3x_2 + x_1x_5x_4 + x_2x_6x_4 + x_3x_6x_5), \\
R(x_1^4) &= \frac{1}{6}(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4), R(x_1^5) = \frac{1}{6}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^4 + x_6^5), \\
24R(x_1^3x_2) &= x_1^3x_2 + x_1^3x_3 + x_2^3x_1 + x_2^3x_3 + x_3^3x_1 + x_3^3x_2 + x_1^3x_4 + x_1^3x_5 + x_4^3x_1 + \\
&\quad + x_4^3x_5 + x_5^3x_1 + x_5^3x_4 + x_2^3x_4 + x_2^3x_6 + x_4^3x_2 + x_4^3x_6 + x_6^3x_2 + x_6^3x_4 + x_3^3x_5 + \\
&\quad + x_3^3x_6 + x_5^3x_3 + x_5^3x_6 + x_6^3x_3 + x_6^3x_5.
\end{aligned}$$

Here

$$R = \frac{1}{n!} \sum_{g \in S_n^{(2)}} g$$

is the Reinolds group action averaging operator which is a projector from $\mathbb{K}[\mathcal{V}_n]$ into $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$.

We have that

$$\gamma(R(x_1^5)) = \gamma(R(x_1^4)) = \gamma(R(x_1^3)) = \gamma(R(x_1^2)) = R(x_1), \gamma(R(x_1^2x_2)) = R(x_1x_2).$$

Therefore, the algebra of invariants $\mathbb{K}[\mathcal{V}_4^0]^{S_4^{(2)}}$ of simple graphs on n vertices is generated by the 4 invariants:

$$R(x_1), R(x_1x_6), R(x_1x_2), R(x_1, x_2, x_3).$$

So far, the algebra of invariants $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ is calculated only for $n \leq 5$, see [5].

3. The Poincaré series of the algebra $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$. Let us consider the algebra $\mathbb{K}[\mathcal{V}_n^0]$ as a vector space. Then the following decomposition into the direct sum of its subspaces holds:

$$\mathbb{K}[\mathcal{V}_n^0] = (\mathbb{K}[\mathcal{V}_n^0])_0 + (\mathbb{K}[\mathcal{V}_n^0])_1 + \cdots + (\mathbb{K}[\mathcal{V}_n^0])_m,$$

where $(\mathbb{K}[\mathcal{V}_n^0])_i$ is the vector space generated by the elements

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m = i, \text{ where } \varepsilon_k = 0 \text{ or } \varepsilon_k = 1.$$

Also, for the algebra $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$ the decomposition holds:

$$\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}} = (\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_0 + (\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_1 + \cdots + (\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_m.$$

Since, the Reynolds operator is a projector which save the degree of a polynomial then the component $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ is generated by the following elements

$$R(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}).$$

Particularly, we have that $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_0 = \mathbb{K}$. Also, the component $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_m$ has the dimension 1 and it is generated by the polynomial $R(x_1x_2 \cdots x_m) = x_1x_2 \cdots x_m$.

Let us now give an interpretation of $\dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ in terms of the graph theory. The following important theorem holds.

Theorem 2. *The dimension $\dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ equal to the number of non-isomorphic simple graphs with n vertices and i edges.*

Proof. The $S_n^{(2)}$ -module $(\mathbb{K}[\mathcal{V}_n^0]/I_m)_i$ is generated by the monomials

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m = i, \varepsilon_k = 0 \text{ or } \varepsilon_k = 1.$$

Since the group $S_n^{(2)}$ is finite then the module $(\mathbb{K}[\mathcal{V}_n^0]/I_m)_i$ is decomposed into the direct sum of its irreducible $S_n^{(2)}$ -submodules:

$$(\mathbb{K}[\mathcal{V}_n^0])_i = M_1 \oplus M_2 \oplus \cdots \oplus M_p.$$

Each of these submodules has a basis generated by the monomials of the form $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}$. Let us choose for each M_i the corresponding basis monomials m_1, m_2, \dots, m_p and consider the invariants $R(m_1), R(m_2), \dots, R(m_p)$. By the construction they are different and linearly independent. Therefore the component $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ is the sum of one-dimensional $S_n^{(2)}$ -submodules

$$(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i = \langle R(m_1) \rangle + \langle R(m_2) \rangle + \cdots + \langle R(m_p) \rangle,$$

and $\dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i = p$ for some p . To each of monomial m_1, m_2, \dots, m_p assign a simple graph in the following way: if $m_i = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}$ then the corresponding simple graph has the form

$$G_{m_i} = \varepsilon_1 \mathbf{e}_1 + \varepsilon_2 \mathbf{e}_2 + \cdots + \varepsilon_m \mathbf{e}_m.$$

Since the monomials m_1, m_2, \dots, m_p belong to the different irreducible $S_n^{(2)}$ -modules then G_{m_i} are non-isomorphic and they exhausted all the possible classes of isomorphic classes of isomorphic graphs with n vertices and i edges. \square

Let us recall that the ordinary generating function for the sequence $\dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$

$$\mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}) = \sum_{i=0}^m \dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i \cdot z^i.$$

is called the Poincaré series of the algebra $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$.

The Theorem 2 implies that

$$g_n(z) = \mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}) = \mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i.$$

In the following theorem we derived explicit formulas for the series $\mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ and $\mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})$.

Theorem 3. *Let the group $S_n^{(2)}$ be realized as $m \times m$ matrices. Then*

$$\begin{aligned} (i) \quad \mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i &= \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(\mathbf{1}_m - \alpha \cdot z^2)}{\det(\mathbf{1}_m - \alpha \cdot z)}, \\ (ii) \quad \mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}) &= \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{1}{\det(\mathbf{1}_m - \alpha \cdot z)}. \end{aligned}$$

here $\mathbf{1}_m$ is the unit $m \times m$ matrix.

Proof. (i) The vector space $(\mathbb{K}[\mathcal{V}_n^0])_1$ has the basis $\langle x_1, x_2, \dots, x_m \rangle$ and a permutation $\alpha \in S_n^{(2)}$ acts on the $(\mathbb{K}[\mathcal{V}_n^0])_1$ by permutation of the basis vectors. Denote this linear operator by A_α . Let us expand this operator on the component $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_k$ as endomorphism and denote it by $A_\alpha^{(k)}$. Since $A_\alpha^{(k)}$ is endomorphism and acts as a permutation of the basis vectors of $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_k$ then the action of the operator $A_\alpha^{(k)}$ is defined correctly.

Let a permutation α be written uniquely as a product of disjoint cycles and let $j_i(\alpha)$ be the number of cycles of length i in the disjoint cycle decomposition of α .

Now find the track of the operator $A_\alpha^{(k)}$.

Lemma 1.

$$\text{Tr}(A_\alpha^{(i)}) = \sum_{\beta_1 + 2\beta_2 + \dots + m\beta_m = i} \binom{j_1(\alpha)}{\beta_1} \binom{j_2(\alpha)}{\beta_2} \dots \binom{j_i(\alpha)}{\beta_i}.$$

Proof. Since the operator $A_\alpha^{(i)}$ acts by permutations of the basis vectors of the vector space $(\mathbb{K}[\mathcal{V}_n^0])_i$ that its track equal to the numbers of its fixed point.

For $i = 1$ we have $(\mathbb{K}[\mathcal{V}_n^0])_1 = \langle x_1, x_2, \dots, x_m \rangle$ and $A_\alpha^{(1)}(x_s) = x_{\alpha^{-1}(s)}$. Thus $\text{Tr}(A_\alpha^{(1)}) = j_1(\alpha)$.

For $i = 2$ let us find out the number of fixed points of the operator $A_\alpha^{(2)}$ which acts on the vector space $(\mathbb{K}[\mathcal{V}_n^0])_2$ with the basis vectors $x_i x_j, i < j$. An arbitrary pair of fixed points of the operator A_α form one fixed point for the operator $A_\alpha^{(2)}$. Thus we get $\binom{j_1(\alpha)}{2}$ such points. Also, every transposition define one fixed point. Therefore

$$\text{Tr}(A_\alpha^{(2)}) = \binom{j_1(\alpha)}{2} + j_2(\alpha).$$

All $j_1(\alpha)$ fixed points of the permutation α generates $\binom{j_1(\alpha)}{3}$ fixed points of the operator $A_\alpha^{(3)}$. Every fixed point of A_α together with $j_2(\alpha)$ transposition generate one fixed point of $A_\alpha^{(3)}$. At last, each any of 3-cycle of α generates one fixed point for $A_\alpha^{(3)}$. Then

$$\text{Tr}(A_\alpha^{(3)}) = \binom{j_1(\alpha)}{3} + j_1(\alpha)j_2(\alpha) + j_3(\alpha).$$

Analogously

$$\text{Tr}(A_\alpha^{(4)}) = \binom{\alpha_1}{4} + \binom{\alpha_1}{2}\alpha_2 + \binom{\alpha_2}{2} + \alpha_1\alpha_3 + \alpha_4.$$

In the general case any partition of i

$$\beta_1 + 2\beta_2 + \dots + m\beta_m = i.$$

generates

$$\binom{j_1(\alpha)}{\beta_1} \binom{j_2(\alpha)}{\beta_2} \dots \binom{j_m(\alpha)}{\beta_m}$$

fixed points of the operator $A_\alpha^{(i)}$. Therefore

$$\text{Tr}(A_\alpha^{(i)}) = \sum_{\beta_1 + 2\beta_2 + \dots + m\beta_m = i} \binom{j_1(\alpha)}{\beta_1} \binom{j_2(\alpha)}{\beta_2} \dots \binom{j_m(\alpha)}{\beta_m}.$$

□

Lemma 2.

$$\sum_{i=0}^m \text{Tr}(A_\alpha^{(i)}) z^i = (1+z)^{j_1(\alpha)} (1+z^2)^{j_2(\alpha)} \dots (1+z^m)^{j_m(\alpha)}.$$

Proof. We have

$$\begin{aligned}
\sum_{i=0}^m \text{Tr}(A_\alpha^{(i)}) z^i &= \sum_{\beta_1+2\beta_2+\dots+m\beta_m=i} \binom{j_1(\alpha)}{\beta_1} \binom{j_2(\alpha)}{\beta_2} \dots \binom{j_m(\alpha)}{\beta_m} z^i = \\
&= \sum_{\beta_1+2\beta_2+\dots+m\beta_m=i} \binom{j_1(\alpha)}{\beta_1} \binom{j_2(\alpha)}{\beta_2} \dots \binom{j_i(\alpha)}{\beta_i} z^{\beta_1+2\beta_2+\dots+m\beta_m} = \\
&= \sum_{\beta_1+2\beta_2+\dots+m\beta_m=i} \binom{j_1(\alpha)}{\beta_1} z^{\beta_1} \binom{j_2(\alpha)}{\beta_2} (z^2)^{\beta_2} \dots \binom{j_i(\alpha)}{\beta_i} (z^m)^{\beta_m} = \\
&= \left(\sum_{\beta_1=0}^m \binom{j_1(\alpha)}{\beta_1} z^{\beta_1} \right) \left(\sum_{\beta_2=0}^m \binom{j_2(\alpha)}{\beta_2} (z^2)^{\beta_2} \right) \dots \left(\sum_{\beta_m=0}^m \binom{j_m(\alpha)}{\beta_m} (z^m)^{\beta_m} \right) = \\
&= (1+z)^{j_1(\alpha)} (1+z^2)^{j_2(\alpha)} \dots (1+z^m)^{j_m(\alpha)}.
\end{aligned}$$

□

Lemma 3.

$$\sum_{i=0}^m \text{Tr}(A_\alpha^{(i)}) z^i = \frac{\det(\mathbf{1}_m - A_\alpha \cdot z^2)}{\det(\mathbf{1}_m - A_\alpha \cdot z)}.$$

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the eigenvalues of the operator A_α . Since $A_\alpha^{m!}$ is the identity matrix then all eigenvalues λ_i are roots of unity of orders $j_1(\alpha), j_2(\alpha), \dots, j_m(\alpha)$. Therefore the characteristic polynomial of the operator A_α has the form

$$\det(\mathbf{1}_m - A_\alpha z) = (1 - \lambda_1 z)(1 - \lambda_2 z) \dots (1 - \lambda_n z) = (1 - z)^{j_1(\alpha)} (1 - z^2)^{j_2(\alpha)} \dots (1 - z^m)^{j_m(\alpha)}.$$

Now

$$\begin{aligned}
\sum_{i=0}^m \text{Tr}(A_\alpha^{(i)}) z^i &= (1+z)^{j_1(\alpha)} (1+z^2)^{j_2(\alpha)} \dots (1+z^m)^{j_m(\alpha)} = \\
&= \frac{(1-z^2)^{j_1(\alpha)}}{(1-z)^{j_1(\alpha)}} \frac{(1-z^4)^{j_2(\alpha)}}{(1-z^2)^{j_2(\alpha)}} \dots \frac{(1-z^{2m})^{j_m(\alpha)}}{(1-z^m)^{j_m(\alpha)}} = \frac{\det(\mathbf{1}_m - A_\alpha \cdot z^2)}{\det(\mathbf{1}_m - A_\alpha \cdot z)}.
\end{aligned}$$

□

Let us find the dimension of $S_n^{(2)}$ -invariant subspace $(\mathbb{K}[\mathcal{V}_n^0])_i$.

Lemma 4.

$$\dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i = \frac{1}{n!} \sum_{\alpha \in G} \text{Tr}(A_\alpha^{(i)}).$$

Proof. The dimension of the subspace $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ is equal to the number of eigenvectors that correspond to the eigenvalue 1 and which are common eigenvectors for all operators $A_\alpha^{(i)}$. Consider the average matrix

$$P^{(i)} = \frac{1}{|G|} \sum_{g \in G} A_\alpha^{(i)}.$$

Since the Reynolds operator is a projector from $(\mathbb{K}[\mathcal{V}_n^0])_i$ into $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ then it has the only eigenvalues 1 and 0. Therefore the dimension of the space $(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i$ is equal to the track of the matrix $P^{(i)}$.

□

Taking into account the lemmas stated above we have

$$\begin{aligned} \mathcal{P}(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}, z) &= \sum_{i=0}^m \dim(\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}})_i z^i = \frac{1}{n!} \sum_{i=0}^m \left(\sum_{\alpha \in S_n^{(2)}} \text{Tr}(A_\alpha^{(i)}) \right) z^i = \\ &= \frac{1}{n!} \sum_{g \in S_n^{(2)}} \left(\sum_{i=0}^m \text{Tr}(A_\alpha^{(i)}) z^i \right) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(\mathbf{1}_m - \alpha \cdot z^2)}{\det(\mathbf{1}_m - \alpha \cdot z)}. \end{aligned}$$

(ii) It is the Molien formula for the Poincaré series of the algebra invariants of the group $S_n^{(2)}$. \square

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